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SOME CHARACTERISTICS OF MENTAL REPRESENTATIONS OF EXPONENTIAL FUNCTIONS – A CASE STUDY WITH PROSPECTIVE TEACHERS

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Starting point is the hypothesis that spontaneous retrieval of knowledge graphs (= concept maps) entails relevant information on the organization, abstractness and the accessibility of mental representations of mathematical objects and allows conclusions on the structure of human long-term memory. In the following case study knowledge graphs on exponential functions from prospective teachers in their third academic year were spontaneously created and recorded, then concluded with interviews in which predominantly the steps of creating the graphic representations were retrieved. Comparison of the concept maps with the interview data revealed structures of the generation process of the knowledge maps, pointing to the meaning creation process of the original mathematical terms.

Objectives and Theoretical Framework

The fact that for the past 10 years – declared by the Congress of the United States as the ‘Decade of the Brain’ – cognitive science has been ‘beginning to understand how the brain works and how it gives rise to the mind’ (Kosslyn &, Koenig, 1995) is not least due to the contributions that the rapidly developing discipline of cognitive neuroscience has been making to this field. As the central concern of didactics is describing, understanding and influencing cognitive processes, we believe that theoretical cognitive research (Sigel, 1999), also in the framework of mathematics didactics, cannot ignore the ‘neuronal dimension’. As this ‘hardware level’ is not accessible for classroom research, we believe that at least the ‘software level’ should be given more attention. This central research dimension entails, amongst other aspects, the question of the *mental representation* of mathematical contents (Davis, 1987).

Our starting point, which brings mental representations immediately into play, is the basic hypothesis that the data collected in our observations entail more or less clear traces of active independent mental constructs from the long-time memory of our research subjects (prospective teachers). If learning is linked to mental rearrangement processes, then we can assume that a ‘new order’ is based on subjective consistent logic at least partially shaped or even controlled by these learning processes. Of course, autonomous processes may also have influenced rearrangement results. Our research interest focuses on these autonomous processes. Which retrieval processes are initiated under spontaneous access appears to us to be an open research question.

The number of contributions in mathematics education on mental representations bas reached a two-figured number, but a uniform definition for mental representations is not in view. After all, context-dependent (see Ball & Bass, 2000) as well as subjective experiences within and outside of mathematics (science, technology, everyday life etc.), which constitute the basis for mental representations, increase the difficulties in striving for a uniform approach. In the last instance individual research questions determine the favored model. Whereas Confrey (1991) views representations more as microstructures, we are more interested in the macrostructure aspect (cf. Pines, 1985). In agreement to this last approach we attempt to understand mental representations of mathematical knowledge as 'network structures' (directed or non-directed graphs) of knowledge that individually guide human information processing (Hasemann &, Mansfield, 1995; Williams, 1998).

Thus, our data gathering process focuses on what may be described in terms of graph nodes together with vertices producing relations between these nodes. We thereby adopt the tool of concept maps developed by Novak & Gowin (1984) (also known in the literature as conceptual graphs), whereby we linguistically prefer the terin 'knowledge graphs' (Zwanefeld, 2000). Novak & Gowin wanted to have an external representation of the way people store information in their minds. In the present study, a knowledge graph is viewed as a structured representation of acquired and (by the investigation) retrieved mathematical knowledge and skills. In the German academic world similar graphic representations (however with more limiting directives) are described as 'mind maps'.

It is clear to us that the structures described here as mental representations are scientific constructs, i.e. models with certain inherent basic assumptions. A description can furthermore only be undertaken indirectly, as we conclude the original representation from the information reproduction. It would be of great interest here to gain data from nuclear magnetic resonance tomography. The quality criteria for our approach are *adequacy* and *viability* of the chosen model, which subsequently justifies the approach or rejects it as unsuitable.

We will, however, extend this static 'two-dimensional' graph structure by a dynamic 'third dimension', namely time. Just as informative as the structure of the graph (the mind map) appears to us the process of its (re-)production in the interview context (when creating the mind map). Indeed our observations point to the assumption that the elements of an individual graph (a person's mind map) belong to different (background-) levels, and that these come into the foreground step by step, i.e. enter the consciousness level of the individual (generic aspect). It is hereby interesting to note whether, and if so, which patterns recognizably appear in these processes.

The chosen theme of *exponential functions* appears to us to be particularly suitable, and even to be of a high degree of relational content for this purpose, and to be an 'ideal critical research site' (Törner, 2001). If we understand the term *exponential function* as the 'conceptual root' of the mind map – as an 'evocational starting point' initiating an individual (re-)production process of a complex knowledge-network –, then this reproduction stemming from the various nodes and vertices – so our hypothesis – follows certain ‘priority rules’. Such rules have already been postulated by Green (1971) in relation to beliefs, where he discusses the functional-generative categories *‘primary versus derivative’ beliefs* as well as the ego involvement categories *‘central versus peripheral’ beliefs*.

Our research objectives within the context of mathematical knowledge can thus be summarized into the following research questions: *What are observable representations? What are patterns of representation* on the background of the reproduction (retrieval) processes? How is then human long-term memory retrieved? It hereby remains an open question insofar this retrieval process is controlled by an object-independent retrieval program, or already contains context-dependent information packages which are unpacked when spontaneously activated (comparable to the memory of a zip file). Through our observation window the arguments seem to point more to the first alternative. The contributions from Sigel (1999) are partially devoted to related questions; only one contribution, however, is specific to mathematics (Lesh, 1999).

Methods of Inquiry – Data Sources

Our data sampling integrates two corresponding data sources from voluntary test persons: *knowledge graphs* and *open interviews* (45–60 minutes, video-taped) on the theme of exponential functions. The survey was undertaken in 2001. Both authors jointly – in the form of a mutual ‘peer debriefing’ – did the work of data collection and evaluation. The interview partners were prospective teachers (upper secondary grades) in their third university year, so that one can principally assume they possess sufficient familiarity with exponential functions. We would like to point out that in Germany it is obligatory for a prospective teacher to study two school subjects at the same level and scope. We will call the 6 students (2. academic discipline in brackets) Andrew (social sciences), Berta (chemistry), Chris (chemistry), David (physics), Ed (physics), and Fred (history).

An open interview style was chosen and carefully designed to avoid the impression of an examination situation. The interview participants were familiar with the concept-mapping method. We presented them for a short time a graphic representation of the Pythagoras theorem. We deliberately did not inform them of the theme of this survey beforehand. After revealing this theme to the students, they then had 10 minutes time to produce (unobserved) a knowledge graph on our theme in question (cf. appendix). As we were concerned with the macro-structuring of this subject matter, we deliberately did not give the candidates ample time to produce their knowledge graphs. We wanted to ensure that each candidate would be able to reconstruct the steps of the knowledge graph generation in the subsequent interview. Moreover, the produced graphic representations would have become too complex and thus their spontaneous quality would have been suppressed.

Following this, in the videotaped Interviews the students were questioned in detail on the creation of their knowledge graphs (cf. the numbering of the nodes in the concept maps). We specifically questioned the students about the aspects: definition and application of exponential functions, their relations to logarithm functions, and their graphic representations – had they not commented on these aspects themselves.

Results

The results presented here focus on the discussion of the graphic representations as a data source; we only marginally refer to the detailed transcriptions of the interviews. It would go beyond the scope of this paper to discuss the complementary, however not contrary, interview results here. We agreed in future to relate the name of the student to the name of the graphic representation produced by that person, as long as this does not lead to any misunderstandings. The mind maps of the six interviewees possess quite a broad spectrum of content richness and structural complexity standing in a certain correlation to the other subjects being studied by the individual students for their teacher degrees. The spectrum reaches from elaborated mind maps (chemistry [Berta], physics [Chris], [Davis]) across reasonably elaborated ones (physics [Ed], social science [Andrew]) down to less elaborated ones (history [Fred]).

As we are interested in spontaneous retrieval processes, we assumed that the global 'locating' of the initially unknown theme is of central relevance.

The Affective Dimension

Immediately after presenting the theme to the students we asked them how they would affectively categorize the theme: Andrew = rather rejective, Berta = positively loaded, Chris = neutral, David = neutral, Ed = "feel insecure" und Fred = neutral, "not very secure". It is surprising how reserved the many different facets this central theme in the academic curriculum are expressed. In particular the students Andrew and Fred, who both cannot relate this theme to their second academic discipline, reacted more rejectively than not.

The 'Home-localization' of the Theme

The spontaneous subject-specific localization of the theme plays a key role, as the following knowledge maps demonstrate. We have to assume that relating a theme to mental information fields activates different retrieval processes. Illuminating was the short question asked in response to our question in the interview of Berta: "May I also use physics examples?" Sources of subjective localization are rather implicit information from the maps as well as verbal statements (e.g., quotations) from the interview. The following table lists these data.

home …	Andrew	Berta	Chris	David	Ed	Fred
map	mathe-matics, calculus	mathe-matics, functional equations	processes of growth in physics	physics	mathe-matics, functions	mathe-matics, Eulerian number
inter-view	mathe-matics	coaching maths stu-dents	everywhere in physics	physics	mathe-matics	mathe-matics

When we speak of home localization we implicitly define in a dual mode a so-called outer location which is not necessarily explicitly mentioned but is nonetheless exactly for this reason present; Fred for example does not like to address aspects from physics: “I am absolutely not very good at physics!”. The inner-outer view (Green employs the terms central and peripheral) underlines the relevance of the home localization, which could be confirmed by the interview statements.

Around the Mathematical World

It has been noted by a number of authors that beyond the dynamics of the conceptual network of mathematics there is a world of stabilized expectations and beliefs which deeply influence the reception as well as the use of mathematical and scientific knowledge (see e.g., Fishbein, 1987). Thus it is fundamental to identify these intuitive forces and to take them into account in analyzing the pattern of the retrieval process. The mathematical worldview of a physics-oriented student will perceive mathematics in a trivial fashion differently than a “pure” mathematician. In particular, mathematical schemata and systematics such as definition, theorem and lemma appear to fade away.

The Pattern of Retrieval

The present six knowledge graphs point to a few retrieval heuristics, which are nonetheless explainable. Basically, when drawing a graph one is generating another node, thus one is probing in a suitable direction towards a new aspect. One of the fundamental principles is the ‘variation of a constant’ – to employ a mathematical principle as a metaphor. A number of approaches appear to be obvious here: First of all there are the completely mathematically neutral classical variation principles: from the *specific to the general* (David) as well as from the *general to the specific* (Ed). Here switching backwards and forwards between the exponential functions (logarithm function) with a random basis a and the classical exponential functions (natural logarithm function) occurs. Alongside these heuristics one can also list variations on the linguistic level when searching for suitable associations (cf. Fred): Here the theme exponential function is also varied over the letter e, synonymous for the Eulerian number. One has the impression of a rather helpless non-directed search. Fred does not leave his mathematics home location; a fruitful link to his parallel academic discipline history cannot occur for content reasons.

More successful, as it is more mathematics-specific, is the *variation on the linguistic level*. Here term representations undergo variations, and formula representations are manipulated. The *variation on the conceptual level* can be viewed as a further development of the aforementioned principle; here we are dealing with associations that can also be understood as dualisms: term representation - graph representation, function - inverse function. Our interviewees represented their knowledge contents - at least in the context of the exponential function - often very clearly in the form of dualisms (graph - term, function - inverse function, application - theory, figurative - formal, approximately - precise, reality - school, learn - forget, know it - look it up). Notably, even dualisms that are in fact symmetric (and when asked, the interviewees are aware of this symmetry) are nonetheless represented as a rule in a hierarchical form. Thus logarithm functions are conceptualized as inverse functions of exponential functions, but not vice-versa. The opposite direction is felt to be 'unnatural': logarithm functions, perceived as 'complicated', are 'reconstructed' from the simpler concept of exponential function by inversion. Other dualisms also appear to be conceptualized as dichotomies with inherent assessment in the forms: 'simple - difficult', 'important - unimportant', 'good - bad' or similar. We tend to interpret these observations in such a manner that representation (even) of mathematical knowledge is 'emotionally charged' (cf. central versus peripheral mathematical beliefs).

Dominance of Context-relevant Aspects

(Aspect: anti-didactic, objective-logical nature of mathematics.)

Characteristic for the (mathematical!) definition of the exponential function is that it fulfills the differential equation $f'=f$. The identity of f to its differentiation is insofar constitutive and unique for this function. It is surprising that this central differentiation aspect in the mind maps is allocated only secondary priority – a tendency more or less strongly supported in the interviews. The same can be said about the interviews, in which conscious mentioning of the $f'=f$-property (if at all) occurs at a very late stage. We explain this fact by noting that the systematics of a function discussion mentions the $f'=f$-property later, and this consequently influences the retrieval process. The logical nature of mathematics has insofar an anti-didactic effect here.

Door-opener Resp. Dead End

We would just like to shortly mention this aspect. Individual nodes of the graphs point to a high networking index, which strikes one's attention immediately. They are equally starting points for (interdisciplinary) associations in various directions, for example the aspect of exponential growth links to physics (see Andrew, Chris, David, Ed), to economics (see Andrew, Berta), to chemistry (see Berta), to geography (see

Chris) and to biology (sec Chris). On the other hand there appear to be nodes that can be called ‘dead ends’, when for example Fred reflects the theme domain under the didactic aspect of how to teach it in schools.

Aspect: Piggyback

Sigel (1999) throws up the question: ‘Does emotion involve a separate representational system or does piggyback affect onto other systems?’ We believe in the productive nature of this piggyback aspect: Our interviews reveal numerous instances of colorful emotionally affected details.

Conclusions

It is apparent to our authors that the observations made here cannot be uncritically generalized without limitations. On the other hand they agree to preceding implicit observations and feature patterns of generalized structures. We consider home localization, which occurs quite spontaneously, to be of decisive importance. The home domain functions as a reference level to which the interviewee always finds his way back, and which also controls retrieval processes to a great extent. In the case of the interviewees, those students akin only to mathematics presented only limited knowledge maps, whereas those students who were science oriented presented more multifaceted aspects. The networking of science contents with mathematical contents and viewpoints, however, had its deficiencies. We have to assume that interdisciplinary linking of content is not automatically executed by the learner but has to be explicitly taught. Moreover, home localization determines the sense-making of mathematical objects und terms (differentiation, growth etc.). It seems as if the university course for mathematics does not take this aspect sufficiently into consideration. Spontaneous reporting of memorized information from the long-term memory does not seem to be determined by the primarily requested content, as if one were to expand a stuffit file. On the contrary: ‘home’-specific routines seem to considerably influence retrieval. Home localization of a theme is insofar a key factor; decisive for home localization is in particular the domain that guarantees sense-making of the mathematical object.

Our observations reveal that the classical way of imparting mathematical knowledge (lectures) does not guarantee the emergence (or construction) of a relevant and sense-making ‘mental network of mathematical knowledge’ in our students – at least in the context of exponential functions. It seems that natural science subjects (chemistry, physics) are allocated specific functionalities here that (regretfully) cannot be mediated solely within mathematics studies. The effects of other subjects upon mathematical knowledge that we observed (a neglected aspect in research) appear to be a rather ambivalent but nonetheless relevant phenomenon for the development of mental representations of mathematical knowledge, which we would like to investigate in further going studies.

From a methodological point of view our survey has demonstrated that research into mind maps only allows correct analyses if in the research context also the production process of each map is illuminated, for instance by a subsequent interview. The mere graph structure of a mind map often leads to improper results, as 'nearness on the paper' may conceal the actual (chronological) distance of mind map elements. While the static graph structure essentially depicts the 'stored representation' of knowledge, the interview offers insight into the dynamics of the 'working memory'.

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